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ASPECTS OF STABILITY OF MULTICRITERIA BOOLEAN LINEAR PROGRAMMING PROBLEM WITH PARAMETRIC OPTIMALITY

V.A. Emelichev[1], Y.V. Nikuln[2,3]

[1]Belarusian State University, Nezavisimisti 4, 220030 Belarus
[2]University of Turku, Vesilinnantie 5, 20014 Finland
[3]Corresponding author: e-mail: yurnik@utu.fi

Abstract. This paper addresses a multicriteria problem of Boolean linear programming with parametric optimality. Parameterizations is introduced by dividing a set of objectives into a family of disjoint subsets, within each Pareto optimality is used to establish dominance between alternatives. The introduction of this principle allows us to connect such classical optimality sets as Pareto and extreme. The parameter space of admissible perturbations in such problem is formed by a set of additive matrices, with arbitrary Hölder's norms specified in the solution and criterion spaces. The lower and upper bounds for the radius of strong stability are obtained with some important properties of attainability as corollaries.

Keywords: post-optimal analysis; multiple criteria; strong stability radius; Boolean linear programming; parametric optimality.

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1. Introduction

Discrete optimization is currently one of the most dynamically developing areas of mathematics. The tasks of discrete mathematics are numerous and diverse. They arise in various fields of mathematics, as well as in economics, technology, and computer science. This circumstance led to a rapid increase in the number of works devoted to the theory and methods of discrete optimization (see, for example, the monograph [27], the review [5] as well as the bibliography therein). The widespread use of discrete optimization models has attracted the attention of many experts to the study of various aspects of stability, as well as the problems of parametric and postoptimal analysis of both scalar (single-criterion) and vector (multicriteria) discrete optimization problems. Despite the abundance of approaches to stability analysis in discrete optimization problems, two main directions can be distinguished: qualitative and quantitative.

In the framework of the qualitative direction, the authors focus on identifying various types of stability of the problem [4,6,15,17,22,23], establishing a relationship between different types of stability [18, 20], as well as on searching and describing the stability region of the optimal solution [28].

The quantitative direction, described in sufficient detail in [14] (see also, [5] and [19]), is associated with obtaining estimates of permissible changes in the initial data of the problem, preserving a certain predetermined property of optimal solutions [2,3,7-13,], and the development of algorithms for calculating these estimates [16,25,26]. The key concept here is the radius of stability, which is understood as the radius of the limit “ball of stability”, i.e. such a neighborhood of the source data in the metric space of the problem parameters that any “perturbed” problem with parameters from this neighborhood has some invariance property with respect to the original problem.

The paper is organized as follows. In section 2, we formulate parametric optimality and introduce basic concepts along with the notation. Section 3 contains some auxiliary statements about norms and two lemmas used later for the proof of the main result. In section 4, we formulate and prove the main result regarding the lower and upper bounds for the strong stability radius. Section 5 lists most important corollaries.

2. Main definitions and problem formulation

Consider a multicriteria Boolean linear programming problem (ILP) in the following formulation. Let $C = [c_{ij}] \in \mathbf{R}^{m\times n}$ be a matrix whose rows are denoted by $C_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in \mathbf{R}^n$, $i \in N_m = \{1,2,\ldots,m\}$, $m \geq 1$. Let $x = (x_1, x_2, \ldots, x_n)^T \in X \subset \mathbf{E}^n$, $n \geq 2, \mathbf{E} = \{0,1\}$, and the number of elements of the set X is finite and greater than one. On the set of (admissible) solutions X, we define a vector linear criterion

$$Cx = (C_1x, C_2x, \ldots, C_mx)^T \to \min_{x\in X}.$$

In the space $\mathbf{R}^k$ of arbitrary dimension $k \in \mathbf{N}$ we introduce a binary relation that generates the Pareto optimality principle [24].

$$y \succ y' \iff y \geq y' \ \& \ y \neq y',$$

where $y = (y_1, y_2, \ldots, y_k)^T \in \mathbf{R}^k$, $y' = (y'_1, y'_2, \ldots, y'_k)^T \in \mathbf{R}^k$.

The symbol $\overline{\succ}$, as usual, denotes the negation of the relation $\succ$.

Let $\emptyset \neq I \subseteq N_m, |I| = v$, and let C_I denote the submatrix of the matrix $C \in \mathbf{R}^{m\times n}$ consisting of rows of this matrix with the numbers of the subset I, i.e.

$$C_I = (C_{i_1}, C_{i_2}, \ldots, C_{i_v})^T, \qquad I = \{i_1, i_2, \ldots, i_v\},$$
$$1 \leq i_1 < i_2 < \ldots < i_v \leq m, \qquad C_I \in \mathbf{R}^{v\times n}.$$

Let $s \in N_m$, and let $N_m = \bigcup_{k\in N_s} I_k$ be a partition of the set N_m into s nonempty sets, i.e. $I_k \neq \emptyset$, $k \in N_s$, and $i \neq j \Rightarrow I_i \cap I_j = \emptyset$. For this partition, we introduce a set of $(I_1, I_2, \ldots, I_s)$-efficient solutions according to the formula:

$$G^m(C, I_1, I_2, \ldots, I_s) = \{x \in X: \ \exists k \in N_s \ \forall x' \in X \ (C_{I_k}x \ \overline{\succ} \ C_{I_k}x')\}.$$

Sometimes for brevity we denote this set by $G^m(C)$.

Obviously, any N_m-efficient solution $x \in G^m(C, N_m)$ $(s = 1)$ is Pareto optimal, i.e. efficient solution to problem (1). Therefore, the set $G^m(C, N_m)$ is the Pareto set [21,24]:

$$P^m(C) = \{x \in X : \ \forall x' \in X \ (Cx \ \overline{\succ} \ Cx')\}.$$

We also use the following set:

$$X(x, C) = \{x' \in X : \ Cx \succ Cx'\},$$

which is a set of solutions $x' \in X$ such that x' dominates x in Pareto sense in problem (1). Therefore,

$$P^m(C) = \{x \in X : \ X(x, C) = \emptyset\}.$$

In the other extreme case, when $s = m$, $G^m(C, \{1\}, \{2\}, \ldots, \{m\})$ is a set of extreme solutions (see e.g. [21]). This set is denoted by $E^m(C)$. Thereby, we have:

$$E^m(C) = \{x \in X : \ \exists k \in N_m \ \forall x' \in X \ (C_k x \ \overline{\succ} \ C_k x')\} =$$
$$\{x \in X : \ \exists k \in N_m \ \forall x' \in X \ (C_k x \leq C_k x')\}.$$

It is easy to see that the set of extreme solutions is composed of the best solutions for each of the m criteria.So, in this context, the parametrization of the optimality principle refers to the introduction of such a characteristic of the binary preference relation that allows us to connect the well-known choice functions, parameterizing them from the Pareto to the extreme.

Denoted by $Z^m(C, I_1, I_2, \ldots, I_s)$, the multicriteria ILP problem consists in finding the set $G^m(C, I_1, I_2, \ldots, I_s)$. Sometimes, for the sake of brevity, we use the notation $Z^m(C)$ for this problem.

It is easy to see that the set $P^1(C) = E^1(C)$ is the set of optimal solutions to the scalar (single-criterion) problem $Z^1(C, N_1)$, where $C \in \mathbf{R}^n$.

For any nonempty subset $I \subseteq N_m$ we introduce the notation

$$P(C_I) = \{x \in X : \ \forall x' \in X \ (C_I x \ \overline{\succ} \ C_I x')\},$$
$$X(x, C_I) = \{x' \in X : \ C_I x \ \succ \ C_I x'\},$$

i.e.

$$P(C_I) = \{x \in X : \ X(x, C_I) = \emptyset\}.$$

Then, by virtue of (2), we obtain

$$G^m(C, I_1, I_2, \ldots, I_s) = \left\{x \in X : \ \exists k \in N_s \ \left(x \in P(C_{I_k})\right)\right\}.$$

Therefore, we have

$$G^m(C, I_1, I_2, \ldots, I_s) = \bigcup_{k \in N_s} P(C_{I_k}), \qquad N_m = \bigcup_{k \in N_s} I_k.$$

It is obvious that all the sets given here are nonempty for any matrix $C \in \mathbf{R}^{m \times n}$.

In the space of solutions $\mathbf{R}^n$, we define an arbitrary Hölder's norm l_p, $p \in [1, \infty]$, i.e. by the norm of a vector $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$ we mean the number

$$\| a \|_p = \begin{cases} \left(\sum\limits_{j \in N_n} |a_j|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|a_j| : \ j \in N_n\} & \text{if } p = \infty. \end{cases}$$

In the space of criteria $\mathbf{R}^m$, we define an arbitrary Hölder's norm l_q, $q \in [1,\infty]$, and $l_p \neq l_q$. By the norm of the matrix $C \in \mathbf{R}^{m\times n}$ with the rows C_i, $i \in N_m$, we mean the norm of a vector whose components are the norms of the rows of the matrix. By that, we have

$$\| C \|_{pq} = \| (\| C_1 \|_p, \| C_2 \|_p, \dots, \| C_m \|_p) \|_q.$$

Obviously,

$$\| C_i \|_p \leq \| C_I \|_{pq} \leq \| C \|_{pq}, i \in I \subseteq N_m.$$

So, it is easy to see that for any $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ with

$$|a_j| = \alpha, \qquad j \in N_n,$$

the following equality holds

$$\| a \|_p = \alpha n^{1/p}$$

for any $p \in [1,\infty]$.

In the solution space $\mathbf{R}^n$ along with the norm l_p, $p \in [1,\infty]$, we will use the conjugate norm l_{p^*}, where the numbers p and p^* are connected, as usual, by the equality

$$\frac{1}{p} + \frac{1}{p^*} = 1,$$

assuming $p^* = 1$ if $p = \infty$, and $p^* = \infty$ if $p = 1$. Therefore, we further suppose that the range of variation of the numbers p and p^* is the closed interval $[1,\infty]$, and the numbers themselves are connected by the above conditions.

Further we use the well-know Hölder's inequality

$$|a^T b| \leq \| a \|_p \| b \|_{p^*}$$

that is true for any two vectors $a = (a_1, a_2, \dots, a_n)^T \in \mathbf{R}^n$ and $b = (b_1, b_2, \dots, b_n)^T \in \mathbf{R}^n$.

Perturbation of the elements of the matrix C is imposed by adding matrices C' from $\mathbf{R}^{m\times n}$ to it. Thus, the perturbed problem $Z^m(C + C')$ has the form

$$(C + C')x \to \min_{x\in X},$$

and the set of its $(I_1, I_2, \dots, I_s)$-efficient solutions is $G^m(C + C', I_1, I_2, \dots, I_s)$.

For an arbitrary number $\varepsilon > 0$, we define the set of perturbing matrices

$$\Omega_{pq}(\varepsilon) = \left\{C' \in \mathbf{R}^{m\times n} : \| C' \|_{pq} < \varepsilon\right\}$$

with rows $C'_i, i \in N_m$.

Following [3,6,9], the *strong stability radius* of the ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$, $m \in \mathbf{N}$, (called T_1-stability radius in the terminology of [9, 10]) is the number

$$\rho = \rho_s^m(p,q) = \begin{cases} \sup \Xi & \text{if } \Xi \neq \emptyset, \\ 0 & \text{if } \Xi = \emptyset, \end{cases}$$

where

$$\Xi = \{\varepsilon > 0 : \ \forall C' \in \Omega_{pq}(\varepsilon) \quad (G^m(C) \cap G^m(C + C') \neq \emptyset)\}.$$

Thus, the strong stability radius of the problem $Z^m(C)$ determines the limit level of perturbations of the elements of the matrix C that preserve optimality of at least one (not necessarily the same) solution of the set $G^m(C)$ of the original problem. For any $C' \in \Omega_{pq}(\varepsilon)$ and $\varepsilon > 0$, it is obvious that $G^m(C) \cap G^m(C + C') \neq \emptyset$ if $G^m(C, I_1, I_2, \dots, I_s) = \mathrm{X}$. Therefore, the problem $Z^m(C)$ with $G^m(C) = \mathrm{X}$ is called *non-trivial*.

3. Lemmas

Before formulating the main result regarding the strong stability radius bounds in the next section, we need to prove two supplementary statements presented in this section as lemma 1 and lemma 2.

Hereinafter, a^+ is a projection of a vector $a = (a_1, a_2, \dots, a_k) \in \mathbf{R}^k$ on a positive orthant, i.e. $a^+ = [a]^+ = (a_1^+, a_2^+, \dots, a_k^+)$, where upperscript $+$ implies positive cut of vector a. That is, we have

$$a_i^+ = [a_i]^+ = \max\{0, a_i\}.$$

Lemma 1. *Given $x^0 \in G^m(C)$, $k \in N_s$ and $\varphi > 0$ such that for any $x \notin G^m(C)$ the inequality*

$$\left\| \left[C_{I_k}(x - x^0) \right]^+ \right\|_q \geq \varphi \parallel x - x^0 \parallel_{p^*} > 0$$

holds. Then the following formula is true:

$$\forall\, x \notin G^m(C)\ \ \forall C' \in \Omega_{pq}(\varphi)\ \ \left((C_{I_k} + C'_{I_k}) x^0 \,\overline{\succ}\, (C_{I_k} + C'_{I_k}) x \right).$$

Proof. Assume there exist *a solution* $\tilde{x} \notin G^m(C)$ and a perturbing matrix $\tilde{C} \in \Omega_{pq}(\varphi)$ such that

$$(C_{I_k} + \tilde{C}_{I_k}) x^0 \succ (C_{I_k} + \tilde{C}_{I_k}) \tilde{x}.$$

Then for any index $i \in I_k$ the following inequality is true:

$$(C_i + \tilde{C}_i) x^0 \geq (C_i + \tilde{C}_i) \tilde{x}.$$

Hence, we have

$$\tilde{C}_i (x^0 - \tilde{x}) \geq C_i (x^0 - \tilde{x}),\ \ i \in I_k\ .$$

From the above we derive

$$\left| \tilde{C}_i (\tilde{x} - x^0) \right| \geq [C_i(\tilde{x} - x^0)]^+,\ \ i \in I_k\ .$$

Taking into consideration Hölder’s inequality (6), we obtain

$$\parallel \tilde{C}_i \parallel_p \parallel \tilde{x} - x^0 \parallel_{p^*} \geq [C_i(\tilde{x} - x^0)]^+,\ \ i \in I_k\ .$$

Due to inequalities (4), we get a contradiction with (7):

$$\varphi \parallel \tilde{x} - x^0 \parallel_{p^*} > \parallel \tilde{C} \parallel_{pq} \parallel \tilde{x} - x^0 \parallel_{p^*} \geq \parallel \tilde{C}_{I_k} \parallel_{pq} \parallel \tilde{x} - x^0 \parallel_{p^*}$$
$$\geq \left\| \left[C_{I_k}(\tilde{x} - x^0) \right]^+ \right\|_q,$$

so formula (8) is valid.

Lemma 2. *Given the formula following formula*

$$\exists x^0 \notin G^m(C)\ \ \exists a \in \mathbf{R}^n\ \ \forall\, x \in G^m(C)\ \ (a^T(x^0 - x) < 0)$$

is true, there exists a non-zero matrix $C^* \in \mathbf{R}^{m \times n}$ such that

$$G^m(C) \cap G^m(C^*) = \emptyset.$$

Proof. Obviously $a \neq \mathbf{0} = (0,0,\dots,0)^T \in \mathbf{R}^m$. Let rows $C_i^*, i \in N_m$ of the matrix $C^* \in \mathbf{R}^{m\times n}$ be defined as:

$$C_i^* = a^T, i \in N_m .$$

Then we get

$$C_i^*(x^0 - x) < 0, i \in N_m .$$

Thus for any index $k \in N_s$, the solution $x \notin P\left(C_{I_k}^*\right)$ if $x \in G^m(C)$. Therefore, we have $x \notin G^m(C^*)$. The last implies

$$G^m(C) \cap G^m(C^*) = \emptyset.$$

4. Bounds

For the multicriteria non-trivial ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$, $m \in \mathbf{N}$, for any $p, q \in [1, \infty]$ and $s \in N_m$ we define:

$$\varphi_s^m(p,q) = \max_{x' \in G^m(C)} \max_{k \in N_s} \min_{x \notin G^m(C)} \frac{\| [C_{I_k}(x - x')]^+ \|_q}{\| x - x' \|_{p^*}}$$

$$\psi_s^m(p,q) = n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin G^m(C)} \max_{x' \in G^m(C)} \max_{k \in N_s} \max_{i \in I_k} \frac{C_i(x - x')}{\| x - x' \|_1}.$$

We are now ready to formulate the main result.

Theorem 1. *For any $m \in \boldsymbol{N}$, $p, q \in [1, \infty]$ and $s \in N_m$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$ has the following lower and upper bounds:*

$$0 < \varphi_s^m(p,q) \le \rho_s^m(p,q) \le \min\{\psi_s^m(p,q), \|C\|_{pq}\}.$$

Proof. Due to (3), the formula is true:

$$\forall\, x' \in G^m(C) \;\; \exists\, k \in N_s \left(x' \in P\left(C_{I_k}\right)\right).$$

Therefore, for any index $k \in N_s$, we get $x \notin P\left(C_{I_k}\right)$ if $x \notin G^m(C)$. From there we conclude that the lower bound is positive, i.e. $\varphi_s^m(p,q) > 0$.

Now we prove that $\rho_s^m(p,q) \ge \varphi_s^m(p,q)$. We chose an arbitrary perturbing matrix $C' \in \mathbf{R}^{m\times n}$ such that it belongs to $\Omega_{pq}\left(\varphi_s^m(p,q)\right)$. In order to prove the lower bound for strong stability radius, it suffices to demonstrate that there exists a solution $x^* \in G^m(C) \cap G^m(C + C')$. According to the definition of the number $\varphi_s^m(p,q)$, there exist a solution $x^0 \in G^m(C)$ and an index $k \in N_s$ such that for any solution $x \notin G^m(C)$ we have:

$$\| [C_{I_k}(x - x^0)]^+ \|_q \ge \varphi_s^m(p,q) \, \| x - x^0 \|_{p^*} > 0.$$

From the above by lemma 1, we get the following formula is true:

$$\forall\, x \notin G^m(C) \;\; \forall C' \in \Omega_{pq}\left(\varphi_s^m(p,q)\right) \left(\left(C_{I_k} + C'_{I_k}\right)x^0 \,\overline{\succ}\, \left(C_{I_k} + C'_{I_k}\right)x\right). \qquad (9)$$

Further, we define a way of selecting a necessary solution $x^* \in G^m(C) \cap G^m(C + C')$, where $C' \in \Omega_{pq}\left(\varphi_s^m(p,q)\right)$. If $x^0 \in G^m(C + C')$, then we select $x^* = x^0$. Otherwise, due to (3) we have $x^0 \notin P(C_{I_k} + C'_{I_k})$. Thus due to the property of outer stability for the Pareto set $P(C_{I_k} + C'_{I_k})$ (see e.g. [26]), we can chose a solution $x^* \in P(C_{I_k} + C'_{I_k})$ such that

$$\left((C_{I_k} + C'_{I_k})x^0 \succ (C_{I_k} + C'_{I_k})\, x^*\right).$$

Taking into account (9), it is easy to derive $x^* \in G^m(C)$. Since $x^* \in G^m(C + C')$, we have just proven that $\rho_s^m(p,q) \geq \varphi_s^m(p,q)$.

Now we prove that $\rho_s^m(p,q) \leq \psi_s^m(p,q)$. First, notice that according to the definition of the number $\psi_s^m(p,q)$, there exists a solution $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \notin G^m(C)$ such that for any solution $x \in G^m(C)$ and any index $k \in N_s$ the following inequality holds:

$$\psi_s^m(p,q) \parallel x^0 - x \parallel_1 \geq n^{\frac{1}{p}} m^{\frac{1}{q}} C_i(x^0 - x),\ i \in I_k. \tag{10}$$

Let $\varepsilon > \psi_s^m(p,q)$. We chose a perturbing matrix $C^0 = [c_{ij}^0] \in \mathbf{R}^{m\times n}$ with rows $C_i^0,\ i \in N_m$ and elements defined as follows:

$$c_{ij}^0 = \begin{cases} -\delta \text{ if } \ i \in N_m \text{ and } x_j^0 = 1, \\ \ \delta \text{ if } \ i \in N_m \text{ and } x_j^0 = 0, \end{cases}$$

where

$$\psi_s^m(p,q) < \delta n^{\frac{1}{p}} m^{\frac{1}{q}} < \varepsilon. \tag{11}$$

Therefore, due to (5) we have

$$\left\|C_i^0\right\|_p = \delta n^{\frac{1}{p}}, i \in N_m,$$

$$\|C^0\|_{pq} = \delta n^{\frac{1}{p}} m^{\frac{1}{q}},$$

$$C^0 \in \Omega_{pq}(\varepsilon).$$

Moreover, the following inequalities are obvious for every index $k \in N_s$:

$$C_i(x^0 - x) = -\delta \parallel x^0 - x \parallel_1 < 0, i \in I_k.$$

Using (10) and (11), we conclude that for any solution $x \in G^m(C)$ and any index $k \in N_s$ the following inequality holds:

$$\left(C_i + C_i^0\right)(x^0 - x) \leq \left(\frac{\psi_s^m(p,q)}{n^{\frac{1}{p}} m^{\frac{1}{q}}} - \delta\right) \parallel x^0 - x \parallel_1 < 0, i \in I_k.$$

Thus for any index $k \in N_s$ we have $x \notin P\left(C_{I_k} + C_{I_k}^0\right)$, and hence, $x \notin G^m(C + C^0)$. Summarizing, for any $\varepsilon > \psi_s^m(p,q)$ there exists the perturbing matrix $C^0 \in \Omega_{pq}(\varepsilon)$ such that $G^m(C) \cap G^m(C + C^0) = \emptyset$, i.e. $\rho_s^m(p,q) < \varepsilon$. Thus, we have just proven that $\rho_s^m(p,q) \leq \psi_s^m(p,q)$.

Finally, we are left to demonstrate that $\rho_s^m(p,q) \leq \|C\|_{pq}$. Let $\varepsilon > \|C\|_{pq}$, $\alpha > 0$ and $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \notin G^m(C)$. We chose a row vector $a = (a_1, a_2, \dots, a_n)$ with elements defined as follows:

$$a_j = \begin{cases} -\alpha \text{ if } \ x_j^0 = 1, \\ \ \alpha \text{ if } \ x_j^0 = 0. \end{cases}$$

Then due to (5) we have

$$\|a\|_p = \alpha n^{\frac{1}{p}}.$$

Moreover, the following inequality is true for any solution $x \in G^m(C)$

$$a(x^0 - x) = -\alpha \parallel x^0 - x \parallel_1 < 0.$$

Therefore due to lemma 2, there exists a non-zero matrix $C^* \in \mathbf{R}^{m\times n}$ such that

$$G^m(C) \cap G^m(C^*) = \emptyset. \qquad (12)$$

Further, we consider a perturbing matrix $C^0 \in \mathbf{R}^{m\times n}$ defined as:

$$C^0 = \eta C^* - C,$$

where $0 < \eta < \frac{\varepsilon - \|C\|_{pq}}{\|C^*\|_{pq}}$.

Then we easily derive

$$\|C^0\|_{pq} = \|\eta C^* - C\|_{pq} \le \eta \|C^*\|_{pq} + \|C\|_{pq} < \varepsilon.$$

Therefore due to (12) we obtain

$$\forall \varepsilon > \|C\|_{pq} \ \ \exists C^0 \in \Omega_{pq}(\varepsilon) \ \ (G^m(C) \cap G^m(C + C^0) = \emptyset).$$

Thus, $\rho_s^m(p,q) < \varepsilon$ for any $\varepsilon > \|C\|_{pq}$. Hence, $\rho_s^m(p,q) \le \|C\|_{pq}$.

5. Corollaries

From theorem 1 we get the following results.

Corollary 1. Let $p = q = \infty$. *For any $m \in \boldsymbol{N}$, and $s \in N_m$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, I_1, I_2, \dots, I_s)$ has the following lower and upper bounds:*

$$0 < \max_{x' \in G^m(C)} \max_{k \in N_s} \min_{x \notin G^m(C)} \max_{i \in I_k} \frac{C_i(x - x')}{\parallel x - x' \parallel_1} \le \rho_s^m(\infty, \infty)$$

$$\le \min_{x \notin G^m(C)} \max_{x' \in G^m(C)} \max_{k \in N_s} \max_{i \in I_k} \frac{C_i(x - x')}{\parallel x - x' \parallel_1}.$$

Corollary 2 [9]. *If s=1, then for any $m \in \boldsymbol{N}$, and for any $p, q \in [1, \infty]$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, N_m)$ of finding the Pareto set $P^m(C)$ has the following lower and upper bounds:*

$$0 < \max_{x' \in P^m(C)} \min_{x \notin P^m(C)} \frac{\|[C(x - x')]^+\|_q}{\|x - x'\|_{p^*}} \le \rho_1^m(p,q) \le$$

$$n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin P^m(C)} \max_{x' \in P^m(C)} \max_{i \in I_k} \frac{C_i(x - x')}{\|x - x'\|_1}.$$

Corollary 3 [1]. *If s=m, then for any $m \in \boldsymbol{N}$, and for any $p, q \in [1, \infty]$, the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, \{1\}, \{2\}, \dots, \{m\})$ of finding the extreme set $E^m(C)$ has the following lower and upper bounds:*

$$0 < \max_{x' \in E^m(C)} \max_{i \in N_m} \min_{x \notin E^m(C)} \frac{C_i(x - x')}{\|x - x'\|_{p^*}} \le \rho_m^m(p,q) \le$$

$$n^{\frac{1}{p}} m^{\frac{1}{q}} \min_{x \notin E^m(C)} \max_{i \in N_m} \max_{x' \in E^m(C)} \frac{C_i(x - x')}{\|x - x'\|_1}.$$

From corollary 2 we have the following result illustrating the fact that the bounds $\varphi_1^m(p,q)$ and $\psi_1^m(p,q)$ are attainable for $p = q = \infty$.

Corollary 4. *If $P^m(C) = \{x^0\}$, then the strong stability radius of the multicriteria non-trivial ILP problem $Z^m(C, N_m)$ of finding the Pareto set $P^m(C)$ is expressed by the formula:*

$$\rho_1^m(\infty,\infty) = \varphi_1^m(\infty,\infty) = \psi_1^m(\infty,\infty) = \min_{x\in X\backslash\{x'\}} \max_{i\in N_m} \frac{C_i(x-x^0)}{\|x-x^0\|_1}.$$

Now we formulate and prove one (a bit more general) result concerning attainability of the upper bound.

Corollary 5. *For any $m \in \boldsymbol{N}$ and any $p,q \in [1,\infty]$, there exists a class of multicriteria non-trivial ILP problem $Z^m(C, N_m)$ of finding the Pareto set $P^m(C)$ such that the strong stability radius is expressed by formula:*

$$\rho_1^m(p,q) = \|C\|_{pq}\,. \qquad (13)$$

Proof. Let $X = \{x^0, x^1, x^2, \dots, x^n\}$, where $x^0 = (0,0,\dots,0)^T \in E^n$, $x^j = e^j, j \in N_n$. Here e^j is a basic column-vector in $\mathbf{R}^n$. Let $C^* = [c_{ij}^*] \in \mathbf{R}^{m\times n}$ be a matrix with negative elements such that $P^m(C^*) = X\backslash\{x^0\}$. In order to make the solution x^0 a a unique Pareto optimal in the perturbed problem $Z^m(C^* + C', N_m)$, $C' = [c'_{ij}] \in \mathbf{R}^{m\times n}$, we have to demand for any solution $x \neq x^0$ the following inequalities to be held:

$$(C^* + C')x^0 \,\overline{>}\, (C^* + C')x.$$

Therefore, $C'x \geq -C^*x$. So, we get $c'_{ij} \geq c^*_{ij}, i \in N_n, j \in N_m$, i.e. $\|C'\|_{pq} \geq \|C^*\|_{pq}$. Thus, we obtain $\rho_1^m(p,q) \geq \|C^*\|_{pq}$. Taking into account theorem 1, we get (13) is true. □

The following below results specify classes of scalar problems with attainable bounds.

Corollary 6. *For any $p,q \in [1,\infty]$, there exists a class of multicriteria non-trivial ILP problem $Z^1(C, N_1), C \in \mathbf{R}^n$, such that the strong stability radius is expressed by formula:*

$$\rho_1^1(p,q) = \psi_1^1(p,q) = \|C\|_{pq}\,.$$

Proof. Let $X = \{x^0, x^1, x^2, \dots, x^n\}$ be same set as in the proof of previous corollary. Let $C = (-\alpha, -\alpha, \dots, -\alpha) \in \mathbf{R}^n, \alpha > 0$. Then we have Cx^0=0, Cx^j=$-\alpha$, $j \in N_n$, $x^0 \notin P^1(C)$, $x^j \in P^1(C), j \in N_n$, and

$$\psi_1^1(p,q) = \|C\|_{pq} = n^{\frac{1}{p}}\alpha.$$

We introduce a perturbing row $C' = (c'_1, c'_2, \dots, c'_n)$ such that $C' \in \Omega_{pq}\left(n^{\frac{1}{p}}\alpha\right)$, i.e. $\|C'\|_{pq} \leq n^{\frac{1}{p}}\alpha$. Proving by contradiction it is easy to show that there exists an index $l \in N_n$ such that $|c'_l| < \alpha$. This yields

$$(C + C')(x^0 - x^l) = \alpha - c'_l > 0,$$

i.e. $x^0 \notin P^1(C + C')$ for any perturbing $C' \in \Omega_{pq}\left(\psi_1^1(p,q)\right)$. Since $x^0 \notin P^1(C)$, we get

$$\rho_1^1(p,q) \geq \psi_1^1(p,q).$$

Taking into account theorem 1and equalities (14), we obtain:

$$\rho_1^1(p,q) = \psi_1^1(p,q) = \|C\|_{pq} = n^{\frac{1}{p}}\alpha$$

Finally, we show that in scalar case all the bounds specified in theorem 1 can be attainable.

Corollary 7. *For any $p,q \in [1,\infty]$, there exists a class of multicriteria non-trivial ILP problem $Z^1(C,N_1), C \in \mathbf{R}^n$, such that the strong stability radius is expressed by formula:*

$$\rho_1^1(p,q) = \varphi_1^1(p,q) = \psi_1^1(p,q) = \|C\|_{pq} .$$

Proof. Let $X = \{x^0, x^1\}$, where $x^0 = (0,0,\dots,0)^T \in E^n$ and $x^1 = (1,1,\dots,1)^T \in E^n$. Let $C = (1,1,\dots,1) \in \mathbf{R}^n$. Then we have $x^0 \in P^1(C),\ x^1 \notin P^1(C)$. Recaling that $\frac{1}{p} + \frac{1}{p^*} = 1$, we obtain:

$$\rho_1^1(p,q) = \varphi_1^1(p,q) = \psi_1^1(p,q) = \|C\|_{pq} = n^{\frac{1}{p}}.$$

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